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Analytical Solutions of the Gravitational Field Equations in de Sitter and Anti-de Sitter Spacetimes

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Abstract The generalized Laplace partial differential equation, describing gravitational fields, is investigated in de Sitter spacetime from several metric approaches—such as the Riemann, Beltrami, Börner-Dürr, and Prasad metrics—and analytical solutions of the derived Riccati radial differential equations are explicitly obtained. All angular differential equations trivially have solutions given by the spherical harmonics and all radial differential equations can be written as Riccati ordinary differential equations, which analytical solutions involve hypergeometric and Bessel functions. In particular, the radial differential equations predict the behavior of the gravitational field in de Sitter and anti-de Sitter spacetimes, and can shed new light on the investigations of quasinormal modes of perturbations of electromagnetic and gravitational fields in black hole neighborhood. The discussion concerning the geometry of de Sitter and anti-de Sitter spacetimes is not complete without mentioning how the wave equation behaves on such a background. It will prove convenient to begin with a discussion of the Laplace equation on hyperbolic space, partly since this is of interest in itself and also because the wave equation can be investigated by means of an analytic continuation from the hyperbolic space. We also solve the Laplace equation associated to the Prasad metric. After introducing the so called internal and external spaces—corresponding to the symmetry groups SO(3,2) and SO(4,1) respectively—we show that both radial differential equations can be led to Riccati ordinary differential equations, which solutions are given in terms of associated Legendre functions. For the Prasad metric with the radius of the universe independent of the parametrization, the internal and external metrics are shown to be of AdS-Schwarzschild-like type, and also the radial field equations arising are shown to be equivalent to Riccati equations whose solutions can be written in terms of generalized Laguerre polynomials and hypergeometric confluent functions.

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1 Introduction

de Sitter and anti-de Sitter spacetimes represent respectively the homogeneous spaces $dS_{4,1} = SO(4,1)/SO(3,1)$ and $AdS_{3,2} = SO(3,2)/SO(3,1)$, and they also correspond to solutions of Einstein equation with cosmological constant $\Lambda = \pm 3/R^2$ $(R > 0)$, and curvature given by the components $R_{\mu\nu} = \Lambda g_{\mu\nu}$ of the Ricci tensor. They present respectively the topology $S^3 \times \mathbb{R}$ and $S^1 \times \mathbb{R}^3$. When the cosmological constant Λ vanishes, absence of gravitation corresponds to Minkowski spacetime, a solution of the sourceless Einstein's equation. Its isometry transformations are those of the Poincaré group, which is the group governing the kinematics of special relativity. For a non-vanishing Λ, however, Minkowski is no longer a solution of the corresponding Λ modified Einstein equation, and in this case, if we interpret Λ as a purely geometric entity, absence of gravitation turns out to be represented by the de Sitter spacetime [1–4].

de Sitter (dS) and anti-de Sitter (AdS) spacetimes are suitable geometric arenas to investigate conformal field theories [5], braneworld scenarios [6], and superstrings [7]. Since de Sitter group is the maximal inner group contained in the conformal group, many physical theories can formulated in dS and AdS scenarios. The maximally compact subgroup of SO(3, 2), the symmetry group of the $dS_{3,2}$ spacetime, is $SO(3) \times SO(2)$, which is two-fold covered by SU(2)×U(1). In this sense, the Lie algebra $\mathfrak{so}(3) \times \mathfrak{so}(2) \simeq \mathfrak{su}(2) \times \mathfrak{u}(1)$—associated with the maximal compact subgroup $SO(3) \times SO(2)$ in the $dS_{3,2}$ spacetime—brings naturally all the information necessary to formally establish the electroweak model formalism in a natural manner. It can therefore be used as a wider alternative formalism to describe the Glashow-Weinberg-Salam model of electroweak interactions [8] in the context of anti-de Sitter geometry, since the gauge group $SU(2) \times U(1)$ is related to the isospin and weak hypercharge of elementary particles. The group SO(3, 2) is also a dynamical group associated to the *Zitterbewegung* [9]. dS and AdS spacetimes also allow exact solutions of the field equations and the symmetry group SO(4, 1) is used to classify physical states [10–12].

In this paper we present and discuss Laplace partial differential equation (LPDE)—emulated by the Hogde-de Rham operator—in these spacetimes. In particular, we emphasize dS and AdS spacetime, from where Minkowski spacetime appears by constructing homeomorphisms from dS, via, e.g., stereographic projections, obtaining the Riemann [12], Börner-Dürr [13, 14] and Beltrami [12] metrics, among other possible metrics. In these cases, we explicitly solve the Laplace differential equation in dS and AdS and for the first time, up to our knowledge we solve the respective radial equations—that are indeed Riccati differential equations. Introducing suitable change of variables, all radial differential equations can be led to Riccati differential equations, and here we present their explicit analytical solutions. Motivated by the original Arcidiacono's approach, the translations in a Minkowskian spacetime can be thought as rotations in the symmetry group SO(4, 1), and can be regarded in both internal and external de Sitter hyperspaces, by associating with each one the symmetry groups SO(4, 1) and SO(3, 2), respectively. After parametrizing these spacetimes, we again solve LPDE, as an alternative method presented by the former projective metrics.

This paper is organized as follows: in Sect. 2 we present Riemann metric derivation of dS homeomorphic projection in Minkowski spacetime and solve LPDE using the Laplace-Beltrami operator associated with the Riemann metric—in this case the radial equation leads to a Riccati differential equation, which explicit solution is derived. In Sect. 3 Beltrami metric is derived by means of a stereographic projection of the center of dS on a tangent Minkowski spacetime at the south pole, and the corresponding equations—coming from LPDE—gives respectively the spherical harmonics and a Riccati equation (corresponding to the radial equation), which solution is obtained analytically in terms of hypergeometric functions. In Sect. 4 the Börner-Dürr metric is derived and an analogous procedure to Sect. 3 is used, but now the solutions of Riccati differential equation is given in terms of Bessel functions of first and second kind. In Sect. 5 the Prasad metric is investigated in terms of a suitable parametrization of internal and external dS spacetimes. The resulting LPDE is lead to a radial equation, which itself leads to a Riccati differential equation—having as solutions associated Legendre functions of the first and second kind. In Sect. 6, after deriving AdS-Schwarzschild black hole metric from a suitable parametrization of internal and external dS spacetimes, we investigate the resulting equations and their analytical solutions.

Hereon we denote $\partial_a = \frac{\partial}{\partial a}$ and ∇^2 denotes the Laplace-Beltrami operator

$$|g|^{-1/2}\partial_a\big(|g|^{1/2}g^{ab}\partial_b\big), \tag{1}$$

where $a, b = 1, \ldots, n$ and $|g|$ denotes the determinant of the metric tensor components matrix g^{ab} associated with the metric g. In a given chart, g can be written as $g = g_{ab}dx^a \otimes dx^b$. Also, we use in the article Einstein summation convention.

2 Riemann Metric Approach

In this section we omit the time coordinate without loss of generality, and Minkowski spacetime is lead to the Euclidean E_3 space. The stereographic projection of dS on 3-dimensional Euclidean space E_3, which can be seen as a tangent space to dS, originates the Riemann metric g_R

$$g_R = \frac{4\mathfrak{R}^2}{4\mathfrak{R}^2+\rho^2}(d\rho\otimes d\rho + \rho^2 d\theta\otimes d\theta + \rho^2\sin^2\theta d\varphi\otimes d\varphi), \tag{2}$$

where (ρ, θ, φ) are spherical coordinates in E_3. Hereon $\mathfrak{R}$ denotes the radius of dS. Substituting the metric above in the formula for the Laplace-Beltrami operator acting on a scalar field $\Phi : E^3 \to \mathbb{R}$ we obtain from the LPDE $\nabla^2\Phi = 0$, after the separation of variables given by $\Phi(\rho, \theta, \varphi) = R(\rho)Y(\theta, \varphi)$, the equations

$$\big(4\mathfrak{R}^2+\rho^2\big)^{1/2}\frac{d}{d\rho}\left(\frac{\rho^2}{(4\mathfrak{R}^2+\rho^2)^{1/2}}\frac{dR}{d\rho}\right) = kR, \tag{3}$$

and

$$\csc\theta\,\partial_\theta(\sin\theta\partial_\theta Y) + \csc^2\theta\,\partial^2_{\varphi\varphi}Y + kY = 0, \tag{4}$$

where[1] $k = \ell(\ell+1)$, $\ell = 0, 1, 2, \ldots$. Equation (4) has the well known spherical harmonics $Y_{\ell,m}(\theta, \varphi)$ as solutions. The radial differential equation can be written as

$$\rho^2\frac{d^2R}{d\rho^2} + \left(2\rho - \frac{\rho^3}{4\mathfrak{R}^2+\rho^2}\right)\frac{dR}{d\rho} - \ell(\ell+1)R = 0 \tag{5}$$

[1] The separation constant k can be identified to $\ell(\ell+1)$ only after the solution of the angular part of the decomposition is obtained.

and by means of the change of variable $\frac{1}{R}\frac{dR}{d\rho} = \eta = \eta(\rho)$ we can write

$$\rho^2\eta' = -\rho^2\eta^2 - \big(2\rho - \rho^3(4\mathfrak{R}^2 + \rho^2)^{-1}\big)\eta + \ell(\ell+1), \tag{6}$$

which is a Riccati differential equation. Defining $\zeta = (9 + 4\ell + 4\ell^2)^{1/2}$ and $\xi = (1 + \ell + \ell^2)^{1/2}$, the general solution of (6) is given in terms of the hypergeometric function ${}_2F_1$ by

$$\begin{aligned}
\eta(\rho) = {} & C\alpha(\rho)^{-1+\frac{1}{2}(-3+\zeta)}\mathfrak{R}^{\frac{1}{2}}(-3+\zeta)(3-\zeta)\rho^{-1+\frac{1}{2}(-3+\zeta)} \\
& \times {}_2F_1\left(\frac{1}{4} - \frac{1}{2}\xi - \frac{1}{4}\zeta, \frac{1}{4} + \frac{1}{2}\xi - \frac{1}{4}\zeta; 1 - \frac{1}{2}\zeta; -\frac{\rho^2}{4\mathfrak{R}^2}\right) \\
& - C\left(1 - \frac{1}{2}\zeta\right)^{-1}(\alpha(\rho)^{-1+\frac{1}{2}(-3+\zeta)})\mathfrak{R}^{-2+\frac{1}{2}(-3+\zeta)}\vartheta\rho^{1+\frac{1}{2}(3-\zeta)} \\
& \times {}_2F_1\left(\frac{5}{4} - \frac{1}{2}\xi - \frac{1}{4}\zeta, \frac{5}{4} + \frac{1}{2}\xi - \frac{1}{4}\zeta; 2 - \frac{1}{2}\zeta; -\frac{\rho^2}{4\mathfrak{R}^2}\right) \\
& + \alpha(\rho)^{-1+\frac{1}{2}(-3+\zeta)}C\mathfrak{R}(-3-\zeta)\alpha(\rho)^{-1+\frac{1}{2}(-3-\zeta)}\rho^{-1+\frac{1}{2}(3+\zeta)} \\
& \times {}_2F_1\left(\frac{1}{4} - \frac{1}{2}\xi + \frac{1}{4}\zeta \frac{1}{4} + \frac{1}{2}\xi + \frac{1}{4}\zeta; 1 + \frac{1}{2}\zeta; -\frac{\rho^2}{4\mathfrak{R}^2}\right) \\
& - C\left(1 + \frac{1}{2}\zeta\right)^{-1}(^{-1+\frac{1}{2}(-3-\zeta)})\mathfrak{R}^{-2+\frac{1}{2}(-3-\zeta)}\vartheta\rho^{1+\frac{1}{2}(3+\zeta)} \\
& \times {}_2F_1\left(\frac{5}{4} - \frac{1}{2}\xi + \frac{1}{4}\zeta, \frac{5}{4} + \frac{1}{2}\xi + \frac{1}{4}\zeta; 2 + \frac{1}{2}\zeta; -\frac{\rho^2}{4\mathfrak{R}^2}\right),
\end{aligned}$$

where C is a integration constant, $\vartheta = ((\frac{1}{4} - \frac{1}{4}\zeta)^2 - \frac{1}{2}\xi^2)$ and

$$\begin{aligned}
\alpha(\rho) := {} & \frac{1}{2}(-3+\zeta)\mathfrak{R}^{\frac{1}{2}}(-3+\zeta)\rho^{\frac{1}{2}(3-\zeta)} \\
& \times {}_2F_1\left(\frac{1}{4} - \frac{1}{2}\xi - \frac{1}{4}\zeta, \frac{1}{4} + \frac{1}{2}\xi - \frac{1}{4}\zeta; 1 - \frac{1}{2}\zeta; -\frac{\rho^2}{4\mathfrak{R}^2}\right) \\
& -2^{\frac{1}{2}(-3-\zeta)}\mathfrak{R}^{\frac{1}{2}}(-3-\zeta)\rho^{\frac{1}{2}(3+\zeta)} \\
& \times {}_2F_1\left(\frac{1}{4} - \frac{1}{2}\xi + \frac{1}{4}\zeta, \frac{1}{4} + \frac{1}{2}\xi + \frac{1}{4}\zeta; 1 + \frac{1}{2}\zeta; -\frac{\rho^2}{4\mathfrak{R}^2}\right).
\end{aligned} \tag{7}$$

3 Beltrami Metric

Given coordinates ξ^a ($a = 0, 1, \ldots, 4$) in dS and the relative coordinates x^μ ($\mu = 0, 1, 2, 3$) in Minkowski spacetime $\mathbb{R}^{1,3}$, the Beltrami metric can be obtained from a stereographic projection of the center of dS on a tangent Minkowski spacetime $\mathbb{R}^{1,3}$ at the south pole $(0, 0, 0, 0, -\mathfrak{R})$, given by $\xi^4 = \frac{\mathfrak{R}}{A}$ and $\xi^\mu = \frac{x^\mu}{A}$, where $A = (1 + \frac{\rho^2}{\mathfrak{R}^2})^{1/2}$, $\rho^2 \equiv x^\mu x_\mu = -x_0^2 + x_1^2 + x_2^2 + x_3^2$ and $\mathfrak{R}$ is the radius of dS.

To a metric in dS there corresponds the Beltrami—projective—metric g_B defined on Minkowski spacetime associated with the same metric, given by

$$g_B = d\xi^a \otimes d\xi_a = A^{-4}\big(A^2 e^\mu \otimes e_\mu - 2\mathfrak{R}^{-2}(x_\mu dx_\nu) \otimes (x^\mu dx^\nu)\big), \tag{8}$$

where $e^\mu := x^\mu dx^\mu$. The associated Laplace differential equation is expressed as

$$\nabla^2\Phi = \frac{(\mathfrak{R}^2+\rho^2)^2}{\mathfrak{R}^4\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\Phi}{\partial\rho}\right) + \frac{(\mathfrak{R}^2+\rho^2)}{\mathfrak{R}^2\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{(\mathfrak{R}^2+\rho^2)}{\rho^2\mathfrak{R}^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} = 0.$$

After separation of variables, the angular differential equation has the same solution as the Riemann metric and the radial differential equation is given by

$$(\mathfrak{R}^2+\rho^2)\frac{\rho^2}{\mathfrak{R}^2}\frac{d^2R}{d\rho^2} + \frac{2\rho}{\mathfrak{R}^2}(\mathfrak{R}^2+\rho^2)\frac{dR}{d\rho} - \ell(\ell+1)R = 0. \tag{9}$$

Introducing the same change of variable done in the case of Riemannian metric, $\mu(\rho) = \frac{1}{R}\frac{dR}{d\rho}$, we can write (9) as

$$\rho^2\mu' = -\mu^2\rho^2 - 2\rho\mu + \ell(\ell+1)\mathfrak{R}^2(\mathfrak{R}^2+\rho^2)^{-1}, \tag{10}$$

which is a Riccati differential equation with solutions given by

$$\begin{aligned}\mu(\rho) &= C\xi^{-1}(\rho)\mathfrak{R}^{\ell+1}(1+\ell)\rho^{-2-\ell}{}_2F_1\left(-\frac{1}{2}(1+\ell)-\frac{\ell}{2};\frac{1}{2}-\ell;-\frac{\rho^2}{\mathfrak{R}^2}\right)\\ &\quad + C\xi^{-1}(\rho)\frac{1+\ell}{1-2\ell}\mathfrak{R}^{\ell-1}\rho^{-\ell}{}_2F_1\left(-\frac{1}{2}(1-\ell),1-\frac{\ell}{2};\frac{3}{2}-\ell;-\frac{\rho^2}{\mathfrak{R}^2}\right)\\ &\quad + C\xi^{-1}(\rho)\mathfrak{R}^{-\ell}\rho^{-\ell+1}{}_2F_1\left(-\frac{1}{2}(1+\ell),\frac{1}{2};\frac{3}{2}+\ell;-\frac{\rho^2}{\mathfrak{R}^2}\right)\\ &\quad + C\xi^{-1}(\rho)\frac{1+\ell}{3+\ell}\mathfrak{R}^{-2-\ell}\rho^{\ell+1}{}_2F_1\left(\frac{1}{2}(3+\ell),1+\frac{\ell}{2};\frac{5}{2}+\ell;-\frac{\rho^2}{\mathfrak{R}^2}\right),\end{aligned} \tag{11}$$

where

$$\begin{aligned}\xi(\rho) &= -\mathfrak{R}^{\ell+1}\rho^{-\ell-1}{}_2F_1\left(-\frac{1}{2}(1+\ell),-\frac{1}{2};\frac{1}{2}-\ell;-\frac{\rho^2}{\mathfrak{R}^2}\right)\\ &\quad - \mathfrak{R}^{-\ell}\rho^{\ell}{}_2F_1\left(\frac{1}{2}(1+\ell),\frac{1}{2};\frac{3}{2}+\ell;-\frac{\rho^2}{\mathfrak{R}^2}\right)\end{aligned} \tag{12}$$

and C is an integration constant. The term ${}_2F_1$ denotes the hypergeometric function.

4 Börner-Dürr Metric

This metric is obtained by the parametrization $\xi^a\eta_{ab}\xi^b = -\mathfrak{R}^2$, with $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1,-1)$, $a,b = 0,\dots,4$ with conformal coordinates $\{x^\mu\}$. It can be realized via a stereographic projection from the north pole $(0,0,0,0,\mathfrak{R})$ into a tangent spacetime at the south pole $(0,0,0,0,-\mathfrak{R})$, accomplished by

$$\xi^\mu = \zeta(x^2)x^\mu, \qquad \xi^4 = -\mathfrak{R}\zeta(x^2)\left(1+\frac{\rho^2}{4\mathfrak{R}^2}\right),$$

where $\zeta(\rho^2) = (1-\frac{x^2}{4\mathfrak{R}^2})^{-1}$, $\rho^2 = x^\mu x_\mu$.

Substituting the metric in Laplace differential equation we obtain

$$\nabla^2\Phi = -\zeta(x^2)\big(\rho^{-2}\partial_\rho(\rho^2\partial_\rho\Phi) + \rho^{-2}\sin\theta(\partial_\theta(\sin\theta\partial_\theta\Phi) + \csc^2\theta) + \partial_{tt}\Phi\big) = 0.$$

Using the method of separation of variables—which leads us to write $\Phi(\rho, t, \theta, \phi) = A(\rho, t)Y(\theta, \phi)$ we obtain the same angular differential equation, although the former equations are partial differential ones and can be written as follows:

$$\partial_\rho(\rho^2\partial_\rho A) + \rho^2\partial_{tt}A = -kA, \tag{13}$$

where k is a constant. Now, introducing $A(\rho, t) = R(\rho)T(t)$ and substituting in (13) we obtain a radial equation

$$\frac{1}{R}\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \frac{k}{\rho^2} = a^2, \tag{14}$$

where a^2 is a constant. Introducing the same change of variable for $R(\rho)$ it follows that

$$\mu'(\rho) = -\mu^2(\rho) - 2\rho^{-1}\mu(\rho) - k\rho^{-2} + a^2, \tag{15}$$

which is also a Riccati equation, with solutions

$$\begin{aligned}\mu(\rho) &= \big[\sqrt{a}\rho(I_{(-\mathfrak{e}_-)}(\sqrt{a}\rho) + \sqrt{a}\rho\, I_{\mathfrak{e}_+}(\sqrt{a}\rho)) - 3I_{\mathfrak{d}}(\sqrt{a}\rho)\big]\chi^{-1}(\rho) \\ &\quad + \big[(\sqrt{a}\rho(I_{(-\mathfrak{e}_+)}(\sqrt{a}\rho)C + \sqrt{a}\rho I_{\mathfrak{e}_-}(\sqrt{a}\rho))C - 3I_{-\mathfrak{d}}(\sqrt{a}\rho))C\big]\chi^{-1}(\rho),\end{aligned} \tag{16}$$

where C is a integration constant, $\mathfrak{d} := \frac{1}{2}\sqrt{9-4k}$, $\mathfrak{e}_\pm := 1 \pm \frac{1}{2}\sqrt{9-4k}$. Also,

$$\chi(\rho) = \rho\big(I_{\mathfrak{d}}(\sqrt{a}\rho) + C\, I_{-\mathfrak{d}}(\sqrt{a}\rho)\big). \tag{17}$$

$I_\nu(z)$ denotes the modified Bessel functions of first kind.

5 Prasad Metric

In this section we investigate the topologically unequivalent spaces proposed by Prasad [10], related to de Sitter and anti-de Sitter spacetimes. In the formalism developed by Prasad, de Sitter and anti-de Sitter spacetimes represents an external and a internal space of his formalism, and to each of these spaces there corresponds the symmetry groups SO(4,1) and SO(3,2) respectively.

5.1 dS Spacetime

This space has constant curvature $1/\mathfrak{R}^2$ and has the following quadratic form associated with:

$$\mathfrak{R}^2 = (x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2 + (x_5)^2.$$

A possible parametrization for the above quadratic form is given by

$$x_1 = \mathfrak{R}\sin\chi\,\sin\theta\cos\varphi\cosh t,$$

$$x_2 = \mathfrak{R}\sin\chi\,\sin\theta\sin\varphi\cosh t,$$

$$x_3 = \Re \sin\chi \cos\theta \cosh t,$$
$$x_4 = \Re \sinh t,$$
$$x_5 = \Re \cos\chi \cosh t,$$

where the variable χ is defined by $\Re^2 \sin^2\chi = (x_1)^2 + (x_2)^2 + (x_3)^2$, allowing us to write the metric as

$$g_+ = \Re^2 \cosh^2 t\big[d\chi \otimes d\chi + \sin^2\chi (d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi)\big] - \Re^2 dt \otimes dt, \tag{18}$$

LPDE reads

$$(\Re^2 \cosh^2 t \sin^2\chi)\nabla^2\Phi = \operatorname{sech} t \sin^2\chi\, \partial_t(\cosh^3 t\, \partial_t\Phi) - \partial_t(\sin^2\chi\, \partial_\chi\Phi) + \csc\theta\, \partial_\theta(\sin\theta\, \partial_\theta\Phi) + \csc^2\theta\, \partial_{\varphi\varphi}\Phi = 0,$$

and after writing the scalar field Φ as $\Phi(\chi, t, \theta, \varphi) = \Gamma(\chi, t)Y(\theta, \varphi)$ we can separate LPDE in two partial differential equations: one of these has the solution known yet (spherical harmonics) while the another is

$$\operatorname{sech} t\, \partial_t(\cosh^3 t\, \partial_t\Gamma) - \csc^2\chi\, \partial_\chi(\sin^2\chi\, \partial_\chi\Gamma) - \ell(\ell+1)\csc^2\chi\, \Gamma = 0, \tag{19}$$

with $\ell = 0, 1, 2, 3\ldots$. Now write $\Gamma(\chi, t) = \Pi(\chi)\Omega(t)$ and suppose that the separation constant is α^2, corresponding to eigenvalue spectrum of Gegenbauer equation. Then, (19) can be separated in two ordinary differential equations:

$$\frac{1}{\sin^2\chi}\frac{d}{d\chi}\left(\sin^2\chi \frac{d\Pi}{d\chi}\right) + \frac{\ell(\ell+1)}{\sin^2\chi}\Pi - \alpha^2\Pi = 0 \tag{20}$$

and

$$\frac{1}{\cosh t}\frac{d}{dt}\left(\cosh^3 t \frac{d\Omega}{dt}\right) - \alpha^2\Omega = 0. \tag{21}$$

We emphasize (20), namely, the radial one, which can be written as

$$\frac{d^2\Pi}{d\chi^2} + 2\cot\chi \frac{d\Pi}{d\chi} + \frac{\ell(\ell+1)}{\sin^2\chi}\Pi - \alpha^2\Pi = 0. \tag{22}$$

Introducing the variable $j \equiv \cos\chi$, we obtain

$$\frac{d^2\Pi}{dj^2}(1-j^2) - 3j\frac{d\Pi}{dj} - \alpha^2\Pi + \frac{\ell(\ell+1)}{1-j^2}\Pi = 0 \tag{23}$$

and replacing $\frac{1}{\Pi}\frac{d\Pi}{dj} = \mu(j)$, we can rewrite (23) as

$$\mu'(j) = -\mu^2(j) + \frac{1}{1-j^2}\left(3j\mu(j) + \alpha^2 - \frac{\ell(\ell+1)}{1-j^2}\right), \tag{24}$$

which is a Riccati equation with solutions given by

$$(j^2-1)^{1/4}\mu(j) = \big[E(jP_\varsigma^\vartheta(j) - 2\varsigma j P_\varsigma^\vartheta(j) + 2(\varsigma+\vartheta)P_{\varsigma-1}^\vartheta(j))\big]\phi^{-1}(j) + \big[(jQ_\varsigma^\vartheta(j) - 2\varsigma j Q_\varsigma^\vartheta(j) + 2(\varsigma+\vartheta)Q_{\varsigma-1}^\vartheta(j))\big]\phi^{-1}(j), \tag{25}$$

where $\varsigma = \frac{1}{2}(2\sqrt{1-\alpha^2}-1)$, $\vartheta = \frac{1}{2}\sqrt{1-4\ell-4\ell^2}$, E denotes a integration constant, and

$$\phi(j) = (j^2-1)^{-1/4}\big(C P_\varsigma^\vartheta(j) + Q_\varsigma^\vartheta(j)\big). \tag{26}$$

Here $P_\varsigma^\vartheta(j)$ and $Q_\varsigma^\vartheta(j)$ denote respectively associated Legendre function of the first and second kind.

5.2 AdS Spacetime

This space has a negative constant curvature $-1/\mathfrak{R}^2$ and can be characterized by the symmetry group SO(3,2), which is associated with the quadratic form

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_6^2 = \mathfrak{R}^2.$$

Prasad proposed [10] the following parametrization with nonstatic coordinates:

$$\begin{aligned} x_1 &= \mathfrak{R}\sinh\chi\,\sin\theta\cos\varphi\cos t, \\ x_2 &= \mathfrak{R}\sinh\chi\,\sin\theta\sin\varphi\cos t, \\ x_3 &= \mathfrak{R}\sinh\chi\,\cos\theta\cos t, \\ x_4 &= \mathfrak{R}\sin t, \\ x_6 &= \mathfrak{R}\cosh\chi\,\cos t. \end{aligned}$$

Then, we can write the metric as follows:

$$g_- = \mathfrak{R}^2\cos^2 t\big[d\chi\otimes d\chi + \sinh^2\chi(d\theta\otimes d\theta + \sin^2\theta d\varphi\otimes d\varphi)\big] - \mathfrak{R}^2 dt\otimes dt. \tag{27}$$

The generalized LPDE can be written as

$$\begin{aligned} \mathfrak{R}^2\cos^2 t\sinh^2\chi\nabla^2\Phi &= -\sec t\sin^2\chi\,\partial_t(\cos^3 t\,\partial_t\Phi) + \partial_\chi(\sinh^2\chi\,\partial_\chi\Phi) \\ &\quad + \csc\theta\,\partial_\theta(\sin\theta\,\partial_\theta\Phi) + \csc^2\theta\,\partial_{\varphi\varphi}\Phi = 0. \end{aligned} \tag{28}$$

Equation (28) can be separated in radial-temporal and angular equations:

$$-\frac{1}{\mho}\frac{\sinh^2\chi}{\cos t}\partial_t(\cos^3 t\,\partial_t\mho) + \frac{1}{\mho}\partial_\chi(\sinh^2\chi\,\partial_\chi\mho) = \ell(\ell+1), \tag{29}$$

where $\ell = 0, 1, 2, \ldots$. It is also worthwhile to mention here that angular equation has spherical harmonics as solutions, and by defining $\mho(\chi, t) = \Xi(\chi)\Psi(t)$ we get two ordinary differential equations

$$\frac{1}{\Psi}\frac{1}{\cos t}\frac{d}{dt}\left(\cos^3 t\frac{d\Psi}{dt}\right) = \beta^2 \tag{30}$$

and

$$\frac{1}{\Xi}\frac{1}{\sinh^2\chi}\frac{d}{d\chi}\left(\sinh^2\chi\frac{d\Xi}{d\chi}\right) + \frac{\ell(\ell+1)}{\sinh^2\chi} = -\beta^2, \tag{31}$$

where β^2 is the separation constant—indeed $\beta = 0, 1, 2, 3, \ldots$ corresponds to the eingenvalue spectrum of Gegenbauer equation. Introducing $m = \cosh\chi$ and defining $\mu(m) =$

$\frac{1}{\Xi}\frac{d\Xi(m)}{dm}$ we can write (31) as

$$\mu'(m) = -\mu^2(m) + \frac{1}{1-m^2}\left(3m\mu(m) + \beta^2 + \frac{\ell(\ell+1)}{1-m^2}\right), \tag{32}$$

which is also a Riccati equation. There is not in general an analytical solution for this equation, but at the fundamental state where $\ell = 0$, the solution is given by

$$\mu(m) = \frac{(-(\gamma\sqrt{1-m^2} + Cm)\cos(\gamma \mathrm{asin} m) + (C\gamma\sqrt{1-m^2} - m)\sin(\gamma \mathrm{asin} m))}{(m^2-1)(C\cos(\gamma \mathrm{asin} m) + \sin(\gamma \mathrm{asin} m))}, \tag{33}$$

where C is a separation constant and $\gamma = \sqrt{1-B^2}$.

6 AdS-Schwarzschild-Like Black Hole Metric

Using Cartesian coordinates, we can express [10] the internal and external metrics,

$$\begin{aligned} g_+ &= dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3 - dx_4 \otimes dx_4 + dx_5 \otimes dx_5, \\ g_- &= dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3 - dx_4 \otimes dx_4 - dx_6 \otimes dx_6. \end{aligned} \tag{34}$$

Using the transformations

$$\begin{aligned} x_5 \pm x_4 &= \mathfrak{R}_+ \exp(\pm t/\mathfrak{R}_+)(1 - r^2/\mathfrak{R}_+^2)^{1/2}, \\ x_4 \pm i x_6 &= \mathfrak{R}_- \exp(\pm it/\mathfrak{R}_-)(1 + r^2/\mathfrak{R}_-^2)^{1/2}, \end{aligned} \tag{35}$$

we can substitute these expressions in (34) obtaining

$$\begin{aligned} g_\pm = {} & (1 \mp r^2/\mathfrak{R}_\pm{}^2)^{-1}(dr \otimes dr) + r^2 d\theta \otimes d\theta \\ & + r^2 \sin^2\theta d\varphi \otimes d\varphi - (1 \mp r^2/\mathfrak{R}_\pm^2) dt \otimes dt, \end{aligned} \tag{36}$$

LPDE reads

$$\begin{aligned} & r^{-2}\partial_r\left(r^2(1 \mp \Lambda_\pm r^2)\partial_r\psi\right) + r^{-2}\csc^2\theta\,\partial_\theta(\sin\theta\,\partial_\theta\psi) \\ & + r^{-2}\csc^2\theta\,\partial^2_{\varphi\varphi}\psi - \frac{1}{1 \mp \Lambda_\pm r^2}\partial^2\psi_{tt} = 0, \end{aligned} \tag{37}$$

where $\Lambda_\pm = 1/\mathfrak{R}_\pm^2$.

Let $\psi(r,t,\theta,\varphi) = Y(\theta,\varphi)\Gamma(r,t)$. Then we can separate (37) as

$$\frac{1}{\Gamma}\frac{\partial}{\partial r}\left[r^2(1 \mp \Lambda_\pm r^2)\frac{\partial\Gamma}{\partial r}\right] - \frac{1}{\Gamma}\frac{r^2}{(1 \mp \Lambda_\pm r^2)}\frac{\partial^2\Gamma}{\partial t^2} = \ell(\ell+1) \tag{38}$$

and for the non-angular equation we obtain the same equation that we have gotten formerly.

If we put $\Gamma(r,t) = S(r)T(t)$, we obtain from (38) the radial equation[2]

$$\frac{1 \mp \Lambda_\pm r^2}{r^2}\frac{d}{dr}\left[r^2(1 \mp \Lambda_\pm r^2)\frac{dS}{dr}\right] - \frac{\ell(\ell+1)}{r^2}(1 \mp \Lambda_\pm r^2)S = BS, \tag{39}$$

[2] We shall not emphasize the temporal equation.

where B is a constant. With the change of variable $\frac{1}{S}\frac{dS}{dr} = \eta(r)$ we can transform (39) in two Riccati differential equations:

$$\eta' = -\eta^2 - \eta\left(\frac{2}{r} \mp \frac{2\Lambda_\pm r}{1 \mp \Lambda_\pm r^2}\right) + \frac{\ell(\ell+1)}{r^2(1 \mp \Lambda_\pm r^2)} + \frac{B}{1 \mp \Lambda_\pm r^2}. \tag{40}$$

The solutions of (40) are also analytical, involving analytical functions of Laguerre and hypergeometric confluent functions. In the particular case where $\ell = 0$, defining

$$\begin{aligned}
h(r) &= (\Lambda_+ r^2(1+B) - B)^{1/2},\\
\iota(r) &= (1 - 2\Lambda_+ r^2 + \Lambda_+ r^4)^{1/2},\\
g(r) &= \frac{h(r)}{\iota(r)},\\
f(r) &= \exp\left(g(r) + \frac{\Lambda_+ r^2(1-\Lambda_+ r^2)\ln r}{\Lambda_+ r^2 - 1}\right),
\end{aligned} \tag{41}$$

the solution of (40) for the case of Λ_+ is given by

$$\begin{aligned}
\eta(r) &= Cf(r)\big(h(r)\iota(r)\big)^{-1}\left(g(r) + \frac{2\Lambda_+ r^2 - 1}{\Lambda_+ r^2 - 1}\right)U\left(\frac{g(r)(\Lambda_+ r(1-r^2))}{h(r)}, 0, -\frac{2h(r)}{\iota(r)}\right)\\
&\quad + 2C\big(h(r)\iota(r)\big)^{-1}\frac{g(r)(\Lambda_+ r(1-r^2))}{f(r)}U\left(1 + \frac{g(r)(\Lambda_+ r(1-r^2))}{h(r)}, 1, -\frac{2h(r)}{\iota(r)}\right)\\
&\quad + 2\big(\iota(r)\big)^{-1}h(r)f(r)\mathcal{L}\left(-1 - \frac{g(r)(\Lambda_+ r(1-r^2))}{h(r)}, -\frac{4h(r)}{\iota(r)}\right)\\
&\quad + a(r)f(r)\left(g(r) + \frac{2\Lambda_+ r^2 - 1}{\Lambda_+ r^2 - 1}\right)L\left(-\frac{g(r)(\Lambda_+ r(1-r^2))}{h(r)}, -1, -\frac{4h(r)}{\iota(r)}\right),
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
a(r) &:= Cf(r)U\left(\frac{g(r)(\Lambda_+ r(1-r^2))}{h(r)}, 0, -\frac{2h(r)}{\iota(r)}\right)\\
&\quad - f(r)L\left(-\frac{g(r)(\Lambda_+ r(1-r^2))}{h(r)}, -1, -2\frac{2h(r)}{\iota(r)}\right),
\end{aligned} \tag{43}$$

and $L(\cdot,\cdot,\cdot)$, $\mathcal{L}(\cdot,\cdot)$, and $U(\cdot,\cdot,\cdot)$ denotes respectively the Laguerre polynomial, the generalized Laguerre polynomial and the hypergeometric confluent function.

The radial part of the scalar field $S(r)$ goes to infinity as we approximate the black hole horizon, defined in (34). Anyway, the solution obtained in (31) is a particular case where $\ell = 0$.

For some physical insights concerning the solutions obtained, we must point out some important remarks concerning the results we have obtained, in the next section.

7 Concluding Remarks and Outlook

We have presented explicit analytical solutions of generalized Laplace partial differential equation in de Sitter and anti-de Sitter spacetimes, investigated using the Riemann, Beltrami, Börner-Dürr, and Prasad metrics. The respective solutions correspond to the gravitational field behaviour in the respective spacetimes.

While de Sitter spacetime is important because it can describe the early and late time behaviour of the real universe, anti-de Sitter spacetimes are important for an entirely different reason, since—among a plenty of physical reasons—it represents the background in which the holographic principle is best understood, in the sense that AdS enjoys properties which are responsible to make it a natural candidate for a holographic Hamiltonian description [16–19].

As the universe AdS $\times S^5$ is widely used in superstring theory, the possibility to generate a cosmological constant for the resulting lower dimensional Kaluza-Klein type theory arises, in a supersymmetric context. It is well known that black holes presenting Schwarzschild radii larger than the curvature radius are stable solutions of the 10-dimensional Einstein equations [16, 18].[3]

Equation (36) should be generalized as

$$g_\pm = \left(1 \mp r^2/{\mathfrak{R}_\pm}^2 \pm \frac{\alpha G}{r^2\mathfrak{R}_\pm^5}\right)^{-1}(dr\otimes dr) + r^2 d\Omega_8 \otimes d\Omega_8$$
$$- \left(1 \mp r^2/\mathfrak{R}_\pm^2 \pm \frac{\alpha G}{r^2\mathfrak{R}_\pm^5}\right) dt\otimes dt, \tag{44}$$

where α is proportional to the mass of the black hole and G is the 10-dimensional Newton constant, and $d\Omega_8 \otimes d\Omega_8$ denotes the 8-dimensional solid angle associated with the metric.

As classically the LPDE analytical solutions exhibited in this paper are necessary from a formal point of view, the character of scalar fields in AdS spacetimes has importance more explicit when we consider, for instance, bulk quantum field theory in Horava-Witten braneworld scenarios, where fields appears in the supergravity description of the bulk. Such fields are indeed 10-dimensional—different from the 4D quantum fields at the boundary theory. There exists some scalar fields prototypes in AdS backgrounds, and a minimally coupled scalar dilaton field ϕ goes to zero at the boundary of AdS [19]. It is not an easy task to predict the most general behavior of the scalar field which is solution of (42). Indeed, we have just obtained the solution for the scalar field 'fundamental state' corresponding to $\ell = 0$. In a very additional particular case, when the constant $\Lambda_\pm = 0$ and also the separation constant $B = 0$, the solution of (42) is given by $\eta(r) = \frac{1}{2r}(\tan(-\kappa(C + \ln(r))/2)$ where $d = \sqrt{-1 + 4k + 4k^2}$ and C is an integration constant. When $B \neq 0$ the solution involves a combination of Bessel functions.

When the minimally coupled scalar action $J = \int d^5x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is taken into account, by considering the metric in (44), it can be shown that $\phi(r) \sim \aleph/r^4$, where the value of $\aleph$ at the boundary is identified with a local field in the super Yang-Mills theory [19]. It should be desirable in a next paper to analyze the explicit solutions of LPDE in AdS from the point of view of (36) and their physical consequences, which is beyond the scope of the present paper.

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[3] With negative cosmological constant.

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